ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 342 (2008) 989–1002

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Aggregation and stability analysis of nonlinear complex systems

A.Yu. Aleksandrov *, A.V. Platonov

Faculty of Applied Mathematics and Control Processes, St. Petersburg State University, 35 Universitetskij Pr., 198504 Petrodvorets, St. Petersburg, Russia

Received 29 May 2007

Available online 28 December 2007

Submitted by M. Nakao

Abstract

The problem of stability of large-scale systems in critical cases is investigated. New form of aggregation for essentially nonlinear complex systems is suggested. With the help of this form the sufficient conditions of asymptotic stability are determined. The results obtained are used for the stability analysis of complex systems by the nonlinear approximation and for the investigation of absolute stability conditions for a certain class of nonlinear systems.

Keywords: Large-scale systems; Lyapunov functions; Absolute stability; Comparison systems

1. Introduction

The main approach for the stability analysis of nonlinear systems is the Lyapunov direct method (the Lyapunov functions method). By the use of this approach, the conditions of stability for many types of systems were obtained. However, it should be noted that until now there are no general constructive methods for the construction of the Lyapunov functions for nonlinear systems.

This problem is especially difficult for the systems of high dimension (for large-scale or complex systems). Therefore for the stability analysis of such systems the composite-system method usually is used [1,7,22]. This method consists of two steps:

(1) decomposition of complex system into the interconnected subsystems, stability investigation of isolated subsystems and construction of the Lyapunov functions for them;
(2) aggregation of the Lyapunov functions obtained in the one scalar or vector Lyapunov function and construction of an auxiliary system (comparison system) stability or instability of which provides the same property for the initial complex system.

* Corresponding author. Fax: +7 (812) 4287159.
E-mail address: alex@vrm.apmath.spbu.ru (A.Yu. Aleksandrov).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.12.061

There are many approaches for the decomposition and aggregation of large-scale systems [1,4,5,7,8,13,14,16,18,22]. These approaches are especially well developed for the construction of linear comparison systems [1,4,5,16]. However, application of linear comparison systems for the stability analysis of essentially nonlinear complex systems yields, in general, "super-sufficient" stability conditions [1]. Various methods of decomposition and aggregation of large-scale systems based on the construction of nonlinear comparison systems were suggested in [1,7,13,14,22]. In particular, in [7,14] stability conditions of complex systems were obtained in terms of stability criterion of autonomous Wazewskij's systems [15]. Besides, the method of stability analysis was presented for the case when isolated subsystems are neutrally stable [7]. In [22] the method of overlapping decomposition of nonlinear systems was suggested. In the monographs [12,13] the conception of matrix-valued Lyapunov's function was developed for the solution of the wide class of stability theory problems.

Nevertheless, the problem of further development of decomposition and aggregation methods for large-scale systems remains an actual one. Its importance is caused by the fact that the stability conditions of complex systems, obtained by the use of a comparison system, depends on the precision of estimation of the Lyapunov function derivative with respect to the system investigated. Therefore, by means of appropriate choice of aggregation form, one can define more exactly the domain of system parameters values, guaranteeing stability of the programmed motion [7,14].

In the present paper, new aggregation form is suggested. This form is used for the construction of nonlinear comparison systems and for the determination of stability conditions for complex systems in critical cases.

2. Statement of the problem

Consider the system

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(t, \mathbf{x}_i) + \mathbf{Q}_i(t, \mathbf{x}), \quad i = 1, \dots, m. \tag{2.1}$$

Here $\mathbf{x}_i \in \mathbf{R}^{n_i}$, $\mathbf{x} = (\mathbf{x}_1^*, \dots, \mathbf{x}_m^*)^*$, the vector functions $\mathbf{F}_i(t, \mathbf{x}_i)$ and $\mathbf{Q}_i(t, \mathbf{x})$ are defined and continuous in the domain $G = \{(t, \mathbf{x})\colon\ t \geqslant 0,\ \|\mathbf{x}\| < \Delta\}$ ($0 < \Delta \leqslant +\infty$, $\|\cdot\|$ is the Euclidean norm of a vector) and satisfy the conditions $\mathbf{F}_i(t, \mathbf{0}) = \mathbf{0}$, $\mathbf{Q}_i(t, \mathbf{0}) = \mathbf{0}$ for all $t \geqslant 0$. Hence, the system considered has the zero solution.

System (2.1) describes the dynamics of complex system composed of m interconnected subsystems [1,7,22]. The functions $\mathbf{F}_i(t, \mathbf{x}_i)$ define the interior connections of subsystems while the functions $\mathbf{Q}_i(t, \mathbf{x})$ characterize the interaction between the subsystems.

Suppose that the zero solutions of isolated systems

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(t, \mathbf{x}_i), \quad i = 1, \dots, m,$$

are asymptotically stable. We will look for the conditions under which the zero solution of (2.1) is also asymptotically stable.

One of the approaches for the determination of such conditions is based on using of the form of large-scale systems aggregation suggested in [16]. According to this approach, it is assumed that for system (2.1) in the domain G there exist the functions $v_i(t, \mathbf{x}_i)$, $\Omega_i(t, \mathbf{x}_i)$, $i = 1, \dots, m$, possessing the following properties:

(a) functions $v_i(t, \mathbf{x}_i)$, $\Omega_i(t, \mathbf{x}_i)$ are positive definite;
(b) functions $\Omega_i(t, \mathbf{x}_i)$ are continuous, functions $v_i(t, \mathbf{x}_i)$ are continuously differentiable, and $v_i(t, \mathbf{x}_i) \to 0$ as $\|\mathbf{x}_i\| \to 0$ uniformly with respect to $t \geqslant 0$;
(c) the inequalities

$$\dot{v}_i|_{(2.1)} \leqslant \Omega_i(t, \mathbf{x}_i) \sum_{j=1}^{m} a_{ij} \Omega_j(t, \mathbf{x}_j), \quad i = 1, \dots, m,$$

are valid, where a_{ij} are constant coefficients, $a_{ij} \geqslant 0$ for $i \neq j$.

It is known [7,22] that under these assumptions the system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \tag{2.2}$$

where $\mathbf{y} \in \mathbf{R}^m$, $\mathbf{A} = (a_{ij})_{i,j=1}^m$, may be considered as the comparison system for (2.1), and the asymptotic stability of (2.2) implies the same property for the zero solution of (2.1).

To prove this fact, one has to choose the Lyapunov function for the initial complex system in the form

$$V(t,\mathbf{x}) = \sum_{i=1}^{m} \lambda_i v_i(t,\mathbf{x}_i). \tag{2.3}$$

Here $\lambda_1,\dots,\lambda_m$ are positive constants.

Differentiating $V(t,\mathbf{x})$ with respect to (2.1), one gets $\dot V|_{(2.1)} \leqslant W(\boldsymbol{\Omega})$, where

$$W(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega}^*(\mathbf{A}^*\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\mathbf{A})\boldsymbol{\Omega}, \quad \boldsymbol{\Omega} = (\Omega_1,\dots,\Omega_m)^*, \quad \boldsymbol{\Lambda} = \operatorname{diag}\{\lambda_1,\dots,\lambda_m\}.$$

Matrix $\mathbf{A}$ is the Metzler matrix [7,22]. Therefore [22], due to asymptotic stability of (2.2) there exist positive numbers $\lambda_1,\dots,\lambda_m$ such that quadratic form $W(\boldsymbol{\Omega})$ is negative definite. Thus, function $V(t,\mathbf{x})$ satisfies all the assumptions of the Lyapunov asymptotic stability theorem [21].

However, it should be noted that the application of above approach for essentially nonlinear complex systems yields, generally, "super-sufficient" stability conditions. The main goal of the present paper is to generalize the considered aggregation form to the nonlinear case.

3. New form of large-scale systems aggregation

We shall assume that for system (2.1) in the domain G there exist the functions $v_i(t,\mathbf{x}_i)$, $\Omega_i(t,\mathbf{x}_i)$, $i=1,\dots,m$, with the following properties:

(a) functions $v_i(t,\mathbf{x}_i)$, $\Omega_i(t,\mathbf{x}_i)$ are positive definite;
(b) functions $\Omega_i(t,\mathbf{x}_i)$ are continuous, functions $v_i(t,\mathbf{x}_i)$ are continuously differentiable, and $v_i(t,\mathbf{x}_i) \to 0$ as $\|\mathbf{x}_i\| \to 0$ uniformly with respect to $t \geqslant 0$;
(c) the inequalities

$$\dot v_i|_{(2.1)} \leqslant a_i \Omega_i^{\gamma_i+\mu_i} + \Omega_i^{\gamma_i} \sum_{j=1}^{k_i} b_{ij} \Omega_1^{\alpha_{i1}^{(j)}} \dots \Omega_m^{\alpha_{im}^{(j)}}, \quad i=1,\dots,m,$$

are valid, where $a_i < 0$ and $b_{ij} > 0$ are constant coefficients, the exponents $\mu_i > 0$, $\gamma_i \geqslant 0$ and $\alpha_{is}^{(j)} \geqslant 0$ satisfy the conditions

$$\sum_{s=1}^{m} \frac{\alpha_{is}^{(j)}}{\gamma_s + \mu_s} \geqslant \frac{\mu_i}{\gamma_i + \mu_i}, \quad j=1,\dots,k_i,\ i=1,\dots,m. \tag{3.1}$$

Theorem 3.1. *If there exist positive numbers $\theta_1,\dots,\theta_m$ such that*

$$a_i \theta_i^{\mu_i} + \sum_{j=1}^{k_i} b_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} < 0, \quad i=1,\dots,m, \tag{3.2}$$

then the zero solution of (2.1) *is asymptotically stable.*

Proof. Construct the Lyapunov function for system (2.1) in the form (2.3). For all $(t,\mathbf{x}) \in G$ we get

$$\dot V|_{(2.1)} \leqslant \sum_{i=1}^{m} \lambda_i a_i \Omega_i^{\gamma_i+\mu_i} + \sum_{i=1}^{m} \lambda_i \Omega_i^{\gamma_i} \sum_{j=1}^{k_i} b_{ij} \Omega_1^{\alpha_{i1}^{(j)}} \dots \Omega_m^{\alpha_{im}^{(j)}}.$$

Thus, to prove asymptotic stability of the zero solution of (2.1) it is sufficient to show that one can choose positive coefficients $\lambda_1,\dots,\lambda_m$ for the function

$$\widetilde W = \sum_{i=1}^{m} \lambda_i a_i y_i^{\gamma_i+\mu_i} + \sum_{i=1}^{m} \lambda_i y_i^{\gamma_i} \sum_{j=1}^{k_i} b_{ij} y_1^{\alpha_{i1}^{(j)}} \dots y_m^{\alpha_{im}^{(j)}}$$

to be negative definite in the nonnegative cone.

Suppose that for some values of indices i and j the corresponding inequalities in (3.1) are strict. In this case, one can construct, instead of $\widetilde{W}$, a new function $\widehat{W}$ by setting $b_{ij} = 0$ for all such i and j. If there exist positive coefficients $\lambda_1, \dots, \lambda_m$ for which $\widehat{W}$ is negative definite, then for these values of $\lambda_1, \dots, \lambda_m$ function $\widetilde{W}$ possesses the same property [23]. Hence, we may assume, without loss of generality, that all the inequalities in (3.1) turn to equalities.

Consider positive numbers $\theta_1, \dots, \theta_m$, satisfying inequalities (3.2). Let us denote $z_i = y_i/\theta_i$, $\eta_i = \theta_i^{\gamma_i}\lambda_i$, $i = 1, \dots, m$. Then function $\widetilde{W}$ takes the form

$$\widetilde{W} = \sum_{i=1}^{m} \eta_i \hat{a}_i z_i^{\gamma_i+\mu_i} + \sum_{i=1}^{m} \eta_i z_i^{\gamma_i} \sum_{j=1}^{k_i} \hat{b}_{ij} z_1^{\alpha_{i1}^{(j)}} \dots z_m^{\alpha_{im}^{(j)}}.$$

Here $\hat{a}_i = a_i \theta_i^{\mu_i}$, $\hat{b}_{ij} = b_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}}$, and $\hat{a}_i + \sum_{j=1}^{k_i} \hat{b}_{ij} < 0$, $i = 1, \dots, m$.

Let $\mathbf{D} = (d_{ij})_{i,j=1}^{m}$, where

$$d_{ii} = \mu_i \hat{a}_i + \sum_{j=1}^{k_i} \hat{b}_{ij} \alpha_{ii}^{(j)}, \quad d_{is} = \sum_{j=1}^{k_s} \hat{b}_{sj} \alpha_{si}^{(j)} \quad \text{for } i \neq s.$$

Matrix $\mathbf{D}$ is the Metzler matrix.

It can be easily shown that the inequality $\mathbf{D}^*\mathbf{h} < \mathbf{0}$ possesses the solution $\tilde{\mathbf{h}} = (1/(\gamma_1 + \mu_1), \dots, 1/(\gamma_m + \mu_m))^*$. Hence [22], there exists a positive solution $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_m)^*$ for the inequality $\mathbf{D}\eta < \mathbf{0}$.

By the use of Jensen inequality [9], one gets that the relations

$$\widetilde{W} \leqslant \sum_{i=1}^{m} \frac{\tilde{\eta}_i \gamma_i}{\mu_i + \gamma_i} z_i^{\mu_i+\gamma_i} \left(\hat{a}_i + \sum_{j=1}^{k_i} \hat{b}_{ij} \right) + \sum_{i=1}^{m} \frac{z_i^{\mu_i+\gamma_i}}{\mu_i + \gamma_i} \sum_{s=1}^{m} d_{is} \tilde{\eta}_s \leqslant -c \sum_{i=1}^{m} z_i^{\mu_i+\gamma_i}$$

are valid for all $\mathbf{z} \in \mathbf{R}^m$. Here c is a positive constant and $\mathbf{z} = (z_1, \dots, z_m)^*$. $\square$

Remark 3.1. As is obvious from the proof of Theorem 3.1, if all the inequalities in system (3.1) are strict, then the zero solution of (2.1) is asymptotically stable (verification of condition of the existence of positive solution for (3.2) is unnecessary). Moreover, in the case of strict inequalities (3.1), the numbers $\theta_i = \tau^{1/(\gamma_i+\mu_i)}$, $i = 1, \dots, m$, satisfy system (3.2) for all sufficiently small values of $\tau > 0$.

Remark 3.2. In the case where $k_i = m$, $\gamma_i = \mu_i = \alpha_{ij}^{(j)} = 1$, $\alpha_{is}^{(j)} = 0$ for $s \neq j$, $i, j, s = 1, \dots, m$, the aggregation form suggested in the present paper coincides with that one suggested in [16].

Remark 3.3. In the case where $k_i = m$, $\gamma_i = 0$, $\mu_i = \alpha_{ij}^{(j)} = 1$, $\alpha_{is}^{(j)} = 0$ for $s \neq j$, $i, j, s = 1, \dots, m$, the aggregation form suggested in the present paper coincides with that one considered in [7].

Next, let us show that the obtained results can be used for the stability analysis of essentially nonlinear complex systems.

4. Stability of complex systems by nonlinear approximation

Suppose that system (2.1) is of the form

$$\dot{\mathbf{x}}_i = \mathbf{H}_i(\mathbf{x}_i) + \sum_{j=1}^{k_i} \mathbf{R}_{ij}(t, \mathbf{x}), \quad i = 1, \dots, m. \tag{4.1}$$

Here the elements of the vectors $\mathbf{H}_i(\mathbf{x}_i)$ are continuously differentiable homogeneous functions of the orders $\mu_i \geqslant 1$, the vector functions $\mathbf{R}_{ij}(t, \mathbf{x})$ are continuous in the domain G and satisfy the inequalities

$$\left\| \mathbf{R}_{ij}(t, \mathbf{x}) \right\| \leqslant c_{ij} \|\mathbf{x}_1\|^{\alpha_{i1}^{(j)}} \dots \|\mathbf{x}_m\|^{\alpha_{im}^{(j)}}, \quad c_{ij} > 0,\ \alpha_{is}^{(j)} \geqslant 0.$$

We will assume that $\sum_{s=1}^{m} \alpha_{is}^{(j)} > 0$, $j = 1, \dots, k_i$, $i = 1, \dots, m$. Under this assumption, system (4.1) has the zero solution.

In [2,10,20] the conditions for the asymptotic stability of the zero solution of (4.1) were obtained. Let us show that the results of those papers may be strengthened by the use of the aggregation form suggested in the previous section.

Suppose that the zero solutions of the isolated systems

$$\dot{\mathbf{x}}_i = \mathbf{H}_i(\mathbf{x}_i), \quad i = 1, \dots, m, \tag{4.2}$$

are asymptotically stable. In this case, it is known [24] that for systems (4.2) there exist the Lyapunov functions $v_i(\mathbf{x}_i)$, $i = 1, \dots, m$, possessing the following properties:

(a) functions $v_i(\mathbf{x}_i)$ are continuously differentiable for all $\mathbf{x}_i \in \mathbf{R}^{n_i}$;
(b) functions $v_i(\mathbf{x}_i)$ are positive homogeneous of the orders $\gamma_i + 1$;
(c) the inequalities

$$a_{1i}\|\mathbf{x}_i\|^{\gamma_i+1} \leqslant v_i(\mathbf{x}_i) \leqslant a_{2i}\|\mathbf{x}_i\|^{\gamma_i+1},$$
$$\left\|\frac{\partial v_i}{\partial \mathbf{x}_i}\right\| \leqslant a_{3i}\|\mathbf{x}_i\|^{\gamma_i}, \quad \left(\frac{\partial v_i}{\partial \mathbf{x}_i}\right)^* \mathbf{H}_i \leqslant -a_{4i}\|\mathbf{x}_i\|^{\gamma_i+\mu_i}$$

are valid for all $\mathbf{x}_i \in \mathbf{R}^{n_i}$, where $a_{1i}, a_{2i}, a_{3i}, a_{4i}$ are positive constants.

Remark 4.1. While constructing the Lyapunov functions $v_1(\mathbf{x}_1), \dots, v_m(\mathbf{x}_m)$, one may take for $\gamma_1, \dots, \gamma_m$ the arbitrary positive numbers [24].

On differentiating $v_i(\mathbf{x}_i)$ with respect to (4.1), we obtain that the estimations

$$\dot{v}_i|_{(4.1)} \leqslant -a_{4i}\|\mathbf{x}_i\|^{\gamma_i+\mu_i} + a_{3i}\|\mathbf{x}_i\|^{\gamma_i} \sum_{j=1}^{k_i} c_{ij}\|\mathbf{x}_1\|^{\alpha_{i1}^{(j)}} \dots \|\mathbf{x}_m\|^{\alpha_{im}^{(j)}}$$

hold for $(t, \mathbf{x}) \in G$, $i = 1, \dots, m$.

Suppose that for chosen values of the parameters $\gamma_1, \dots, \gamma_m$ conditions (3.1) are fulfilled. Applying Theorem 3.1 (here $\Omega_i(t, \mathbf{x}_i) = \|\mathbf{x}_i\|$, $i = 1, \dots, m$), we get the validity of the following

Theorem 4.1. *If there exist positive numbers $\theta_1, \dots, \theta_m$, satisfying the inequalities*

$$-a_{4i}\theta_i^{\mu_i} + a_{3i}\sum_{j=1}^{k_i} c_{ij}\theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} < 0, \quad i = 1, \dots, m, \tag{4.3}$$

then the zero solution of (4.1) *is asymptotically stable.*

Remark 4.2. Coefficients a_{3i}, a_{4i} in (4.3) depend, in general, on the chosen values of $\gamma_1, \dots, \gamma_m$.

Remark 4.3. In [2,10,20] it was supposed that there exist positive constants $\gamma_1, \dots, \gamma_m$ under which all the inequalities in (3.1) are strict. The approach suggested by A.A. Kosov [10] may be used also in the case where a part (or all) of inequalities in (3.1) turn to equalities. However, for fixed values of $\gamma_1, \dots, \gamma_m$, Theorem 4.1 provides one with the more precise conditions of asymptotic stability in comparison with those obtained in [10]. On the other hand, for the application of Kosov's approach it is not necessary for the parameters $\gamma_1, \dots, \gamma_m$ to satisfy additional restrictions (3.1).

Remark 4.4. In the case where $0 < \mu_i < 1$, $i = 1, \dots, m$, by the similar way the conditions of finite time stability [17] for system (4.1) can be obtained.

5. Criterion for absolute stability

Consider now the system

$$\dot{x}_i = a_i f_i(x_i) + \sum_{j=1}^{k_i} b_{ij} f_1^{\alpha_{i1}^{(j)}}(x_1) \dots f_m^{\alpha_{im}^{(j)}}(x_m), \quad i = 1, \dots, m. \tag{5.1}$$

Here $x_i \in \mathbf{R}^1$, a_i and b_{ij} are constant coefficients, scalar functions $f_i(x_i)$ are defined and continuous for $|x_i| < \Delta$, $0 < \Delta \leqslant +\infty$, and possess the property $x_i f_i(x_i) > 0$ for $x_i \neq 0$, the exponents $\alpha_{is}^{(j)}$ are nonnegative rationals with odd denominators.

System (5.1) is a generalization of the following one

$$\dot{x}_i = \sum_{j=1}^{m} b_{ij} f_j(x_j), \quad i = 1, \dots, m,$$

which is widely used in automatic control systems design [6,11,19].

Let the inequalities $\sum_{s=1}^{m} \alpha_{is}^{(j)} > 0$, $j = 1, \dots, k_i$, $i = 1, \dots, m$, hold. The fulfilment of this assumption provides the existence of the zero solution for system (5.1). Furthermore, we shall suppose that coefficients a_i and b_{ij} satisfy the conditions

$$a_i < 0, \quad b_{ij} > 0, \quad j = 1, \dots, k_i, \ i = 1, \dots, m. \tag{5.2}$$

For instance, inequalities (5.2) are valid if (5.1) is obtained as comparison system for a complex system [1,7,22].

Definition 5.1. System (5.1) is absolutely stable if the zero solution of this system is asymptotically stable for any admissible functions $f_1(x_1), \dots, f_m(x_m)$.

Let us investigate the conditions of absolute stability for (5.1).

In addition, we shall assume that one can choose positive rationals $\gamma_1, \dots, \gamma_m$ with odd numerators and denominators for the inequalities

$$\sum_{s=1}^{m} \frac{\alpha_{is}^{(j)}}{\gamma_s + 1} \geqslant \frac{1}{\gamma_i + 1}, \quad j = 1, \dots, k_i, \ i = 1, \dots, m, \tag{5.3}$$

to be fulfilled.

Definition 5.2. System (5.1) satisfies the Martynyuk–Obolenskij condition [15] (MO-condition) if there exist positive numbers $\theta_1, \dots, \theta_m$ such that

$$a_i \theta_i + \sum_{j=1}^{k_i} b_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} < 0, \quad i = 1, \dots, m. \tag{5.4}$$

Theorem 5.1. *System* (5.1) *is absolutely stable if and only if it satisfies the MO-condition.*

Proof. *Necessity*. Let us note that in the case where $f_i(x_i)$ are nondecreasing functions, (5.1) is Wazewskij's system [7]. In [15] the autonomous Wazewskij's systems were treated. The criterion for the asymptotic stability in the positive cone of the zero solution was obtained. Using this result, we get that the MO-condition is a necessary one for system (5.1) to be absolutely stable.

Sufficiency. Choose the positive rationals $\gamma_1, \dots, \gamma_m$ with odd numerators and denominators, satisfying inequalities (5.3). Consider the functions

$$v_i(x_i) = \int_0^{x_i} f_i^{\gamma_i}(\tau)\, d\tau, \quad i = 1, \dots, m.$$

Functions $v_i(x_i)$ are continuously differentiable and positive definite. On differentiating $v_i(x_i)$ with respect to (5.1), we arrive at

$$\dot{v}_i|_{(5.1)} \leqslant a_i f_i^{\gamma_i+1} + |f_i|^{\gamma_i} \sum_{j=1}^{k_i} b_{ij} |f_1|^{\alpha_{i1}^{(j)}} \ldots |f_m|^{\alpha_{im}^{(j)}}$$

for all $\|\mathbf{x}\| < \Delta$, where $\mathbf{x} = (x_1, \ldots, x_m)^*$. $\square$

Applying Theorem 3.1 (here $\Omega_i(x_i) = |f_i(x_i)|$, $\mu_i = 1$, $i = 1, \ldots, m$), we get that the zero solution of (5.1) is asymptotically stable.

Corollary 5.1. *System* (5.1) *is absolutely stable if and only if for this system there exists the Lyapunov function of the form*

$$V(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \int_0^{x_i} f_i^{\gamma_i}(\tau)\, d\tau, \tag{5.5}$$

satisfying the assumptions of the Lyapunov asymptotic stability theorem. Here $\lambda_1, \ldots, \lambda_m$ are positive coefficients, $\gamma_1, \ldots, \gamma_m$ are positive rationals with odd numerators and denominators.

Remark 5.1. As is obvious from the proof of Theorem 5.1, while constructing the Lyapunov function (5.5), one may take for $\gamma_1, \ldots, \gamma_m$ the arbitrary positive rationals with odd numerators and denominators, satisfying inequalities (5.3).

Corollary 5.2. *Let for system* (5.1) *the MO-condition is fulfilled. If there exist parameters $\gamma_1, \ldots, \gamma_m$ values under which all the inequalities in* (5.3) *turn to equalities, and*

$$\int_0^{x_i} f_i^{\gamma_i}(\tau)\, d\tau \to +\infty \quad \text{as } |x_i| \to \infty,\ i = 1, \ldots, m,$$

then the zero solution of (5.1) *is globally asymptotically stable.*

Remark 5.2. In a similar way, the criterion for absolute stability can be obtained for the case when the inequalities $b_{ij} > 0$ in (5.2) are replaced by those connecting coefficients b_{ij} and a basis $\omega_1, \ldots, \omega_m$ [19]:

$$b_{ij}\omega_i \omega_1^{\alpha_{i1}^{(j)}} \ldots \omega_m^{\alpha_{im}^{(j)}} > 0, \quad j = 1, \ldots, k_i,\ i = 1, \ldots, m. \tag{5.6}$$

Here every constant $\omega_1, \ldots, \omega_m$ takes either of the values $+1$ or -1.

Suppose now that for coefficients b_{ij} there is no such a basis $\omega_1, \ldots, \omega_m$ that inequalities (5.6) are valid. Consider the auxiliary system

$$\dot{x}_i = a_i f_i(x_i) + \sum_{j=1}^{k_i} |b_{ij}| f_1^{\alpha_{i1}^{(j)}}(x_1) \ldots f_m^{\alpha_{im}^{(j)}}(x_m), \quad i = 1, \ldots, m. \tag{5.7}$$

Corollary 5.3. *If system* (5.7) *satisfies the MO-condition, then system* (5.1) *is absolutely stable.*

However, it should be noted that Corollary 5.3 gives one only a sufficient condition for the absolute stability of (5.1).

6. One more approach for the Lyapunov functions construction

Consider again system (5.1) with the coefficients a_i and b_{ij} satisfying inequalities (5.2). Now we shall assume that for the admissible functions $f_i(x_i)$ the following additional conditions are fulfilled: $f_i(x_i)$ are continuously differentiable for $|x_i| < \Delta$, and $f_i'(x_i) > 0$ for $0 < |x_i| < \Delta$, $f_i'(0) \geqslant 0$, $i = 1, \ldots, m$.

Construct the Lyapunov function for system (5.1) in the form

$$V_1(\mathbf{x}) = \max_{i=1,\dots,m} \left(\frac{f_i(x_i)}{\theta_i} \right)^{\gamma_i+1}. \tag{6.1}$$

Here $\theta_1, \dots, \theta_m$ are positive constants and $\gamma_1, \dots, \gamma_m$ are positive rationals with odd numerators and denominators. Function $V_1(\mathbf{x})$ is positive definite and continuous for $\|\mathbf{x}\| < \Delta$.

Denote by $D^+V_1(\mathbf{x})$ the upper right Dini derivative of $V_1(\mathbf{x})$ with respect to (5.1) [21].

Theorem 6.1. *System* (5.1) *is absolutely stable if and only if for this system there exists the Lyapunov function of the form* (6.1) *such that the inequality*

$$D^+V_1(\mathbf{x}) \leqslant W(\mathbf{x})$$

holds for $\|\mathbf{x}\| < H$. Here $H > 0$ is a constant and $W(\mathbf{x})$ is a negative definite function.

Proof. The sufficiency is obvious. Let us prove the necessity.

Suppose that (5.1) is absolutely stable. Then the MO-condition is fulfilled for this system. Let us take for the parameters values $\gamma_1, \dots, \gamma_m$ and $\theta_1, \dots, \theta_m$ of the Lyapunov function (6.1) the solutions of systems (5.3) and (5.4) correspondingly.

Let $0 < H < \Delta$. Consider a solution $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^*$ of (5.1) and a $\hat{t} \geqslant 0$, satisfying the condition $\|\mathbf{x}(\hat{t})\| < H$.

Find

$$B = \max_{i=1,\dots,m} \left(\frac{f_i(x_i(\hat{t}))}{\theta_i} \right)^{\gamma_i+1}.$$

Denote by A such a subset of $\{1, \dots, m\}$ that

$$\left(\frac{f_i(x_i(\hat{t}))}{\theta_i} \right)^{\gamma_i+1} = B \quad \text{for } i \in A, \qquad \left(\frac{f_i(x_i(\hat{t}))}{\theta_i} \right)^{\gamma_i+1} < B \quad \text{for } i \notin A.$$

If positive constant H is sufficiently small, then for every $i \in A$ the relations

$$\begin{aligned} \frac{d}{dt} \left(\left(\frac{f_i(x_i(t))}{\theta_i} \right)^{\gamma_i+1} \right) \bigg|_{t=\hat{t}} &= \frac{\gamma_i+1}{\theta_i^{\gamma_i+1}} f_i^{\gamma_i}\big(x_i(\hat{t})\big) f_i'\big(x_i(\hat{t})\big) \Bigg(a_i f_i\big(x_i(\hat{t})\big) \\ &\quad + \sum_{j=1}^{k_i} b_{ij} f_1^{\alpha_{i1}^{(j)}}\big(x_1(\hat{t})\big) \dots f_m^{\alpha_{im}^{(j)}}\big(x_m(\hat{t})\big) \Bigg) \\ &\leqslant \frac{\gamma_i+1}{\theta_i} B f_i'\big(x_i(\hat{t})\big) \left(a_i\theta_i + \sum_{j=1}^{k_i} b_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} \right) \end{aligned}$$

are valid. Hence,

$$D^+V_1\big(\mathbf{x}(\hat{t})\big) \leqslant B \max_{i \in A} \left\{ \frac{\gamma_i+1}{\theta_i} f_i'\big(x_i(\hat{t})\big) \left(a_i\theta_i + \sum_{j=1}^{k_i} b_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} \right) \right\}.$$

By the use of this estimation, function $W(\mathbf{x})$, satisfying the conditions of the theorem, can be easily constructed. □

Remark 6.1. Constant H and function $W(\mathbf{x})$ in the statement of Theorem 6.1 depend, generally, on the chosen admissible functions $f_1(x_1), \dots, f_m(x_m)$.

Next, let us show that for the solution of some problems the using of the Lyapunov function of the form (6.1) is more effective in comparison with the using of that one constructed by formula (5.5).

Suppose that coefficients a_i and b_{ij} in system (5.1) are functions of t, defined and continuous for all $t \geqslant 0$. Thus, we shall consider the system

$$\dot{x}_i = a_i(t) f_i(x_i) + \sum_{j=1}^{k_i} b_{ij}(t) f_1^{\alpha_{i1}^{(j)}}(x_1) \dots f_m^{\alpha_{im}^{(j)}}(x_m), \quad i = 1, \dots, m. \tag{6.2}$$

Constructing the Lyapunov function for (6.2) in the form (5.5) and applying the approach, suggested in Section 5, one can show the validity of the following

Theorem 6.2. *Let $a_i(t) \leqslant \bar{a}_i$, $|b_{ij}(t)| \leqslant \bar{b}_{ij}$ for all $t \geqslant 0$, where $\bar{a}_i < 0$ and $\bar{b}_{ij} \geqslant 0$ are constants, and there exist positive numbers $\theta_1, \dots, \theta_m$ such that*

$$\bar{a}_i \theta_i + \sum_{j=1}^{k_i} \bar{b}_{ij} \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} < 0, \quad i = 1, \dots, m.$$

Then system (6.2) *is absolutely stable.*

On the other hand, by the use of the Lyapunov function (6.1), we get the validity of the

Theorem 6.3. *Let there exist positive numbers $\eta, \theta_1, \dots, \theta_m$ such that*

$$a_i(t)\theta_i + \sum_{j=1}^{k_i} \left|b_{ij}(t)\right| \theta_1^{\alpha_{i1}^{(j)}} \dots \theta_m^{\alpha_{im}^{(j)}} < -\eta, \quad i = 1, \dots, m,$$

for all $t \geqslant 0$. Then system (6.2) *is absolutely stable.*

In comparison with Theorem 6.2, Theorem 6.3 provides one with the more precise approximation of the domain of absolute stability in parameters space.

Example 6.1. Suppose that system (6.2) is of the form

$$\begin{cases} \dot{x}_1 = (c + 2\sin t) f_1(x_1) + \cos^2 t f_2^3(x_2), \\ \dot{x}_2 = -f_2(x_2) + f_1^{1/3}(x_1), \end{cases} \tag{6.3}$$

where c is a constant.

The condition for the absolute stability of system (6.3) provided by Theorem 6.2 is: $c < -3$, while Theorem 6.3 yields: $c < -2$.

7. Conditions of the existence of positive solutions for the system of inequalities

In the previous sections it was shown that for the construction of the Lyapunov functions for (4.1) and (5.1) one should choose positive numbers $\gamma_1, \dots, \gamma_m$, satisfying systems of inequalities (3.1) and (5.3) respectively.

In the present section, we shall obtain the conditions for the existence of the required values of $\gamma_1, \dots, \gamma_m$, and create an algorithm for finding these numbers.

Consider system (3.1), where $\mu_i > 0$, $\alpha_{is}^{(j)} \geqslant 0$, $j = 1, \dots, k_i$, $i, s = 1, \dots, m$. Denote $h_i = 1/(\gamma_i + \mu_i)$, $i = 1, \dots, m$. Then inequalities (3.1) take the form

$$-\mu_i h_i + \sum_{s=1}^{m} \alpha_{is}^{(j)} h_s \geqslant 0, \quad j = 1, \dots, k_i, \ i = 1, \dots, m. \tag{7.1}$$

Suppose that there exists at least one couple of indices i, j with $j \in \{1, \dots, k_i\}$, $i \in \{1, \dots, m\}$, such that $\mu_i > \alpha_{ii}^{(j)}$. In the opposite case, the problem of the existence of positive solutions for (7.1) is trivial.

Along with (7.1), consider the system

$$-\mu_i h_i + \sum_{s=1}^{m} \alpha_{is}^{(j)} h_s = c_i^{(j)}, \quad j = 1, \ldots, k_i,\ i = 1, \ldots, m, \tag{7.2}$$

where $c_i^{(j)}$ are nonnegative constants. This system may be splitted into m subsystems. Let us apply to (7.2) the modified Gaussian elimination procedure. On the ith step of this procedure each of the equations with negative coefficient of h_i in the ith subsystem is used for the elimination of h_i from the $(i+1)$th, etc., and mth subsystems. This results in a new set of subsystems with (generally) the other number of equations than in the initial system.

One may assume, without loss of generality, that after the application of the above procedure we obtain the system

$$\sum_{s=i}^{m} \beta_{is}^{(j)} h_s = \tilde{c}_i^{(j)}, \quad j = 1, \ldots, q_i,\ i = 1, \ldots, r,$$
$$\sum_{s=r+1}^{m} \beta_{is}^{(j)} h_s = \tilde{c}_i^{(j)}, \quad j = 1, \ldots, q_i,\ i = r+1, \ldots, m. \tag{7.3}$$

Here $1 \leqslant r \leqslant m$; $\tilde{c}_i^{(j)} \geqslant 0$, $j = 1, \ldots, q_i$, $i = 1, \ldots, m$; $\beta_{is}^{(j)} \geqslant 0$ for $s = i+1, \ldots, m$, $j = 1, \ldots, q_i$, $i = 1, \ldots, r$; $\beta_{is}^{(j)} \geqslant 0$ for $s = r+1, \ldots, m$, $j = 1, \ldots, q_i$, $i = r+1, \ldots, m$; for every $i \in \{1, \ldots, r\}$ there exists at least one value of $j \in \{1, \ldots, q_i\}$ such that $\beta_{ii}^{(j)} < 0$.

Theorem 7.1. *System* (7.1) *has a positive solution $\tilde{\mathbf{h}} = (\tilde{h}_1, \ldots, \tilde{h}_m)^*$ if and only if $r < m$, and for all values of $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, q_i\}$, such that $\beta_{ii}^{(j)} < 0$, there exists at least one positive coefficient among $\beta_{ii+1}^{(j)}, \ldots, \beta_{im}^{(j)}$. Moreover, if in any equation of system* (7.3) *there exists at least one positive coefficient, then one can choose positive numbers $\tilde{h}_1, \ldots, \tilde{h}_m$ under which all the inequalities in* (7.1) *become strict.*

Proof. The necessity is obvious. Let us prove the sufficiency.

Consider system (7.3). Let $\tilde{h}_{r+1}, \ldots, \tilde{h}_m$ are arbitrary positive numbers, $B_i = \{j\colon\ j \in \{1, \ldots, q_i\},\ \beta_{ii}^{(j)} < 0\}$,

$$\tilde{h}_i = \min_{j \in B_i} \frac{1}{\beta_{ii}^{(j)}} \left(\varepsilon^{r+1-i} - \sum_{s=i+1}^{m} \beta_{is}^{(j)} \tilde{h}_s \right), \quad i = 1, \ldots, r,$$

where ε is a nonnegative constant. If ε is sufficiently small, then $\tilde{h}_i > 0$, $i = 1, \ldots, m$.

For chosen values of $\tilde{h}_1, \ldots, \tilde{h}_m$ the following conditions are fulfilled:

(1) $\tilde{c}_1^{(j)} = c_1^{(j)}$, $j = 1, \ldots, q_1$, and $q_1 = k_1$;
(2) for every $i \in \{2, \ldots, r\}$ and $j \in \{1, \ldots, k_i\}$ there exists $l \in \{1, \ldots, q_i\}$ such that
$$\tilde{c}_i^{(l)} = c_i^{(j)} + \sum_{s=1}^{i-1} d_{is}^{(l)} \varepsilon^{r+1-s};$$
(3) for every $i \in \{r+1, \ldots, m\}$ and $j \in \{1, \ldots, k_i\}$ there exists $l \in \{1, \ldots, q_i\}$ such that
$$\tilde{c}_i^{(l)} = c_i^{(j)} + \sum_{s=1}^{r} d_{is}^{(l)} \varepsilon^{r+1-s}.$$

Here $d_{is}^{(l)}$ are independent of ε nonnegative coefficients, whose values are determined under the application of the modified Gaussian elimination procedure.

We have $\tilde{c}_i^{(l)} \geqslant \varepsilon^{r+1-i}$ if $\beta_{ii}^{(l)} < 0$, and $\tilde{c}_i^{(l)} \geqslant 0$ for the other values of indices i, l.

If in every equation of system (7.3) there exists at least one positive coefficient $\beta_{is}^{(j)}$, then $\tilde{c}_i^{(l)} > 0$, $l = 1, \ldots, q_i$, $i = 1, \ldots, m$. Therefore, for sufficiently small $\varepsilon > 0$, we get $c_i^{(j)} > 0$, $j = 1, \ldots, k_i$, $i = 1, \ldots, m$. In this case for chosen numbers $\tilde{h}_1, \ldots, \tilde{h}_m$ all the inequalities in (7.1) are strict.

If at least one equation in system (7.3) contains only zero coefficients $\beta_{is}^{(j)}$, then we take $\varepsilon = 0$. In this case for chosen numbers $\tilde{h}_1, \dots, \tilde{h}_m$ a part of inequalities (7.1) turn to equalities. $\square$

Remark 7.1. The proof of Theorem 7.1 contains a constructive algorithm for finding a positive solution for (7.1). Moreover, in the case where $\alpha_{is}^{(j)}$ are nonnegative rationals with odd denominators, using this algorithm, one can choose $\tilde{h}_{r+1}, \dots, \tilde{h}_m$ such that the numbers $\tilde{\gamma}_i = 1/\tilde{h}_i - 1$, $i = 1, \dots, m$, become positive rationals with odd numerators and denominators.

Definition 7.1. We shall say that the $\widehat{\text{MO}}$-condition is fulfilled for system (5.1) if for any $\delta > 0$ there exist positive numbers $\theta_1, \dots, \theta_m$, satisfying inequalities (5.4), such that $\theta_i < \delta$, $i = 1, \dots, m$.

Remark 7.2. Such form of the Martynyuk–Obolenskij condition was used in [3,14].

Corollary 7.1. *If the $\widehat{MO}$-condition is fulfilled for* (5.1)*, then system* (5.3) *possesses a positive solution.*

Proof. Consider the system

$$-h_i + \sum_{s=1}^{m} \alpha_{is}^{(j)} h_s = c_i^{(j)}, \quad j = 1, \dots, k_i,\ i = 1, \dots, m, \tag{7.4}$$

where $c_i^{(j)}$ are nonnegative constants. Let us apply to (7.4) the modified Gaussian elimination procedure. This procedure generates equivalent systems of linear equations with the coefficients changed in the similar way as the orders of $\theta_1, \dots, \theta_m$ under the successive elimination of these variables from (5.4).

Since in any neighborhood of the origin $(\theta_1, \dots, \theta_m)^* = (0, \dots, 0)^*$ there exists a positive solution for system (5.4), the above mentioned procedure reduces (7.4) to the form (7.3) with the coefficients $\beta_{is}^{(j)}$, satisfying the conditions of Theorem 7.1. $\square$

Remark 7.3. By the use of the stability criterion for the autonomous Wazewskij's systems [15], it can be easily shown that if (5.1) is absolutely stable, then the $\widehat{\text{MO}}$-condition is fulfilled for this system. Hence, it turns out the assumption from Section 5, concerning the existence of positive rationals $\gamma_1, \dots, \gamma_m$ with odd numerators and denominators, satisfying inequalities (5.3), was not imposed only by applied technique of the proof of Theorem 5.1. As a matter of fact, this assumption is a necessary condition for (5.1) to be absolutely stable.

8. Stability conditions for the system composed from two interconnected oscillators

Let the system

$$\begin{cases} \ddot{x}_1 + a_1 \dot{x}_1^{\nu_1} + b_1 x_1 = f_1(t, \mathbf{x}, \dot{\mathbf{x}}), \\ \ddot{x}_2 + a_2 \dot{x}_2^{\nu_2} + b_2 x_2 = f_2(t, \mathbf{x}, \dot{\mathbf{x}}) \end{cases} \tag{8.1}$$

be given. Here $x_1, x_2 \in R^1$, $\mathbf{x} = (x_1, x_2)^*$; a_1, a_2, b_1, b_2 are positive constants; $\nu_1 \geqslant 1$ and $\nu_2 \geqslant 1$ are rationals with odd numerators and denominators; the functions $f_1(t, \mathbf{x}, \dot{\mathbf{x}})$ and $f_2(t, \mathbf{x}, \dot{\mathbf{x}})$ are continuous for $t \geqslant 0$, $\|\mathbf{x}\| < \Delta$, $\|\dot{\mathbf{x}}\| < \Delta$ ($\Delta > 0$ is a constant), and satisfy the inequalities

$$\left|f_1(t, \mathbf{x}, \dot{\mathbf{x}})\right| \leqslant \beta_1 \left(x_2^2 + \dot{x}_2^2\right)^{\alpha_1/2}, \qquad \left|f_2(t, \mathbf{x}, \dot{\mathbf{x}})\right| \leqslant \beta_2 \left(x_1^2 + \dot{x}_1^2\right)^{\alpha_2/2},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. Thus, (8.1) possesses the zero solution.

System (8.1) describes the interaction between two nonlinear oscillators

$$\begin{aligned} &\ddot{x}_1 + a_1 \dot{x}_1^{\nu_1} + b_1 x_1 = 0, \\ &\ddot{x}_2 + a_2 \dot{x}_2^{\nu_2} + b_2 x_2 = 0. \end{aligned} \tag{8.2}$$

It is known [6,21] that the zero solutions of isolated Eqs. (8.2) are asymptotically stable. We are looking for the conditions under which the zero solution of interconnected system (8.1) is also asymptotically stable.

Construct the Lyapunov functions in the form

$$v_i = \frac{1}{\gamma_i + 1}\left(b_i \frac{x_i^2}{2} + \frac{\dot{x}_i^2}{2} + \eta_i x_i^{\nu_i} \dot{x}_i\right)^{\gamma_i + 1},$$

where $\eta_i > 0$, $\gamma_i \geqslant 0$, $i = 1, 2$.

On differentiating these functions with respect to (8.1), we get

$$\dot{v}_i|_{(8.1)} = \left(b_i \frac{x_i^2}{2} + \frac{\dot{x}_i^2}{2} + \eta_i x_i^{\nu_i} \dot{x}_i\right)^{\gamma_i} \big(-a_i \dot{x}_i^{\nu_i+1} - \eta_i b_i x_i^{\nu_i+1} + \eta_i \nu_i x_i^{\nu_i-1} \dot{x}_i^2 - \eta_i a_i x_i^{\nu_i} \dot{x}_i^{\nu_i}$$
$$+ \big(\dot{x}_i + \eta_i x_i^{\nu_i}\big) f_i(t, \mathbf{x}, \dot{\mathbf{x}})\big), \quad i = 1, 2.$$

For sufficiently small values of Δ, η_1, η_2, there exist positive numbers c_{ji} ($i = 1, 2,\ j = 1, 2, 3, 4$) such that the estimations

$$c_{1i}\big(x_i^2 + \dot{x}_i^2\big) \leqslant b_i \frac{x_i^2}{2} + \frac{\dot{x}_i^2}{2} + \eta_i x_i^{\nu_i} \dot{x}_i \leqslant c_{2i}\big(x_i^2 + \dot{x}_i^2\big),$$
$$\big|\dot{x}_i + \eta_i x_i^{\nu_i}\big| \leqslant c_{3i}\big(x_i^2 + \dot{x}_i^2\big)^{1/2},$$
$$-a_i \dot{x}_i^{\nu_i+1} - \eta_i b_i x_i^{\nu_i+1} + \eta_i \nu_i x_i^{\nu_i-1} \dot{x}_i^2 - \eta_i a_i x_i^{\nu_i} \dot{x}_i^{\nu_i} \leqslant -c_{4i}\big(x_i^2 + \dot{x}_i^2\big)^{(\nu_i+1)/2}$$

are valid for $\|\mathbf{x}\| < \Delta$, $\|\dot{\mathbf{x}}\| < \Delta$, $i = 1, 2$.

Let us denote $\Omega_i = (x_i^2 + \dot{x}_i^2)^{1/2}$, $i = 1, 2$. Then we obtain the inequalities

$$\dot{v}_1|_{(8.1)} \leqslant -c_{41} c_{11}^{\gamma_1} \Omega_1^{2\gamma_1+\nu_1+1} + \beta_1 c_{31} c_{21}^{\gamma_1} \Omega_1^{2\gamma_1+1} \Omega_2^{\alpha_1},$$
$$\dot{v}_2|_{(8.1)} \leqslant -c_{42} c_{12}^{\gamma_2} \Omega_2^{2\gamma_2+\nu_2+1} + \beta_2 c_{32} c_{22}^{\gamma_2} \Omega_2^{2\gamma_2+1} \Omega_1^{\alpha_2}.$$

In this case, system (3.1) is of the form

$$\begin{cases} \dfrac{\alpha_1}{2\gamma_2 + \nu_2 + 1} \geqslant \dfrac{\nu_1}{2\gamma_1 + \nu_1 + 1}, \\ \dfrac{\alpha_2}{2\gamma_1 + \nu_1 + 1} \geqslant \dfrac{\nu_2}{2\gamma_2 + \nu_2 + 1}. \end{cases} \tag{8.3}$$

If $\alpha_1\alpha_2 > \nu_1\nu_2$, then one can choose the parameters γ_1 and γ_2 values for inequalities (8.3) to be strict.

Assume, next, that $\alpha_1\alpha_2 = \nu_1\nu_2$. Then γ_1 and γ_2 should satisfy the equality

$$2\gamma_2 + \nu_2 + 1 = \frac{\alpha_1}{\nu_1}(2\gamma_1 + \nu_1 + 1). \tag{8.4}$$

Applying Theorem 3.1, we get the validity of the

Theorem 8.1. *The zero solution of* (8.1) *is asymptotically stable if one of the following conditions is fulfilled*:

(a) $\alpha_1\alpha_2 > \nu_1\nu_2$;
(b) $\alpha_1\alpha_2 = \nu_1\nu_2$ *and*

$$\frac{c_{41}}{\beta_1 c_{31}} \left(\frac{c_{42}}{\beta_2 c_{32}}\right)^{\frac{\nu_1}{\alpha_2}} \left(\frac{c_{11}}{c_{21}}\right)^{\gamma_1} \left(\frac{c_{12}}{c_{22}}\right)^{\frac{\gamma_2\nu_1}{\alpha_2}} > 1 \tag{8.5}$$

with γ_1 *and* γ_2 *satisfying* (8.4).

Remark 8.1. If $\alpha_1\alpha_2 = \nu_1\nu_2$, then the approximation of the asymptotic stability domain in parameters space, defined by inequality (8.5), depends on the chosen values of γ_1 and γ_2. It can be easily shown that to obtain the most precise approximation, we should take

$$\gamma_1 = 0, \quad \gamma_2 = (\alpha_1 + \alpha_1/\nu_1 - \nu_2 - 1)/2 \quad \text{if } \alpha_1 + \alpha_1/\nu_1 - \nu_2 - 1 \geqslant 0,$$
$$\gamma_1 = (\alpha_2 + \alpha_2/\nu_2 - \nu_1 - 1)/2, \quad \gamma_2 = 0 \quad \text{if } \alpha_1 + \alpha_1/\nu_1 - \nu_2 - 1 < 0.$$

9. Conclusion

In the present paper, the new form of aggregation of large-scale systems is suggested. This form is a generalization of those considered in [4,5,16]. In comparison with the known results, the approach suggested is more effective in the case when the systems investigated are essentially nonlinear. Using this form, new stability conditions for the certain classes of nonlinear complex systems are obtained.

In particular, the theorem on the stability of large-scale systems by the nonlinear approximation is proved. This theorem, in general, provides one with the more precise stability conditions than those obtained in [2,10,20,23]. Besides, it is shown that this aggregation form can be used for the stability analysis of equilibrium positions of nonlinear mechanical systems. Moreover, the criterion of absolute stability for a class of nonlinear systems is established. It should be noted that this criterion looks similar to that one for the asymptotic stability of autonomous Wazewskij's systems [15]. However, in comparison with the results of [15], in the present paper, it has been proved that the only MO-condition is a sufficient one for the asymptotic stability of the zero solution of (5.1), i.e. the other assumptions from [15] (concerning the uniqueness of solutions, isolation of the equilibrium position at the origin and nondecreasement of the functions $f_i(x_i)$) are redundant.

By the appropriate choice of functions $v_i(t, \mathbf{x}_i)$ and $\Omega_i(t, \mathbf{x}_i)$, $i = 1, \ldots, m$, the approach suggested can be extended to the more wide classes of differential equations systems.

For instance, if the entries of the vectors $\mathbf{H}_i(\mathbf{x}_i) = (H_{i1}(\mathbf{x}_i), \ldots, H_{in_i}(\mathbf{x}_i))^*$ in (4.2) are generally homogeneous functions [23] of the class $(\xi_{i1}, \ldots, \xi_{in_i})$ and of the orders $\sigma_i + \xi_{ij}$ correspondingly, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$, the functions $\Omega_1(\mathbf{x}_1), \ldots, \Omega_m(\mathbf{x}_m)$ should be chosen in the form

$$\Omega_i(\mathbf{x}_i) = \sum_{j=1}^{n_i} |x_{ij}|^{1/\xi_{ij}}, \quad i = 1, \ldots, m.$$

Here $x_{i1}, \ldots, x_{in_i}$ are the entries of the vector $\mathbf{x}_i$. In this case, by analogy with the proof of Theorem 4.1, the stability conditions for large-scale systems by the generally homogeneous approximation can be obtained.

Moreover, the results of this paper can be used for the stability analysis of difference and time-delay systems in critical cases.

References

[1] R.Z. Abdullin, L.Yu. Anapolsky, et al., Vector Lyapunov Functions in Stability Theory, Adv. Ser. Math. Sci. Eng., World Federation Publishers Company, Atlanta, 1996.

[2] A.Yu. Aleksandrov, Stability of complex systems in critical cases, Avtomat. i Telemekh. 9 (2001) 3–13 (in Russian); English translation: Autom. Remote Control 62 (9) (2001) 1397–1406.

[3] A.Yu. Aleksandrov, A.V. Platonov, Construction of Lyapunov's functions for a class of nonlinear systems, Nonlinear Dyn. Syst. Theory 6 (1) (2006) 17–29.

[4] M. Araki, Application of M-matrices to the stability problems of composite dynamical systems, J. Math. Anal. Appl. 52 (2) (1975) 309–321.

[5] M. Araki, Stability of large-scale nonlinear systems—Quadratic-order theory of composite-system method using M-matrices, IEEE Trans. Automat. Control AC-23 (2) (1978) 129–142.

[6] E.A. Barbashin, Introduction to the Theory of Stability, Wolters–Noordhoff Publishing, Groningen, 1970.

[7] Lj.T. Grujic, A.A. Martynyuk, M. Ribbens-Pavella, Large Scale Systems Stability under Structural and Singular Perturbations, Springer-Verlag, Berlin, 1987.

[8] P. Habets, K. Peiffer, Classification of stability-like concepts and their study using vector Lyapunov functions, J. Math. Anal. Appl. 43 (2) (1973) 537–570.

[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1986.

[10] A.A. Kosov, On the stability of complex systems by the nonlinear approximation, Differ. Uravn. 33 (10) (1997) 1432–1434 (in Russian); English translation: Differ. Equ. 33 (10) (1997) 1440–1442.

[11] S. Lefschetz, Stability of Nonlinear Control Systems, Academic Press, New York, 1965.

[12] A.A. Martynyuk, Stability by Liapunov's Matrix Function Method with Applications, Marcel Dekker, New York, 1998.

[13] A.A. Martynyuk, Qualitative Methods in Nonlinear Dynamics: Novel Approaches to Liapunov's Matrix Functions, Marcel Dekker, New York, 2002.

[14] A.A. Martynyuk, On the theory of Lyapunov's direct method, Dokl. Ross. Akad. Nauk 408 (3) (2006) 309–312 (in Russian); English translation: Dokl. Math. 73 (3) (2006) 376–379.

[15] A.A. Martynyuk, A.Yu. Obolenskij, Stability of solutions of autonomous Wazewskij systems, Differ. Uravn. 16 (8) (1980) 1392–1407 (in Russian); English translation: Differ. Equ. 16 (1981) 890–901.

[16] A.N. Michel, Stability analysis of interconnected systems, SIAM J. Control 12 (3) (1974) 554–579.

[17] E. Moulay, W. Perruquetti, Finite time stability and stabilization of a class of continuous systems, J. Math. Anal. Appl. 323 (2006) 1430–1443.
[18] Y. Ohta, D.D. Siljak, Overlapping block diagonal dominance and existence of Liapunov functions, J. Math. Anal. Appl. 112 (2) (1985) 396–410.
[19] S.K. Persidskij, Problem of absolute stability, Avtomat. i Telemekh. 12 (1969) 5–11 (in Russian); English translation: Autom. Remote Control 1969 (1970) 1889–1895.
[20] A.V. Platonov, On stability of complex nonlinear systems, Izv. Ross. Akad. Nauk Teor. Sist. Upr. 4 (2004) 41–46 (in Russian); English translation: J. Comput. Syst. Sci. Int. 43 (4) (2004) 531–536.
[21] N. Rouche, P. Habets, M. Laloy, Stability Theory by Lyapunov's Direct Method, Springer, New York, 1977.
[22] D.D. Siljak, Decentralized Control of Complex Systems, Academic Press, New York, 1991.
[23] V.I. Zubov, Mathematical Methods for the Study of Automatical Control Systems, Pergamon Press/Jerusalem Acad. Press, Oxford/Jerusalem, 1962.
[24] V.I. Zubov, Methods of A.M. Lyapunov and Their Applications, P. Noordhoff Ltd., Groningen, 1964.